Supersymmetry and the Multi-Instanton Measure

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We propose explicit formulae for the integration measure on the moduli space of charge-n ADHM multi-instantons in $N = 1$ and $N = 2$ supersymmetric gauge theories. The form of this measure is fixed by its (super)symmetries as well as the physical requirement of clustering in the limit of large spacetime separation between instantons. We test our proposals against known expressions for $n \leq 2$. Knowledge of the measure for all n allows us to revisit, and strengthen, earlier $N = 2$ results, chiefly: (1) For any number of flavors N_F, we provide a closed formula for $\mathcal{F}_n$, the n-instanton contribution to the Seiberg-Witten prepotential, as a finite-dimensional collective coordinate integral. This amounts to a solution, in quadratures, of the Seiberg-Witten models, without appeal to electric-magnetic duality. (2) In the conformal case $N_F = 4$, this means reducing to quadratures the previously unknown finite renormalization that relates the microscopic and effective coupling constants, τ_{micro} and τ_{eff}. (3) Similar expressions are given for the 4-derivative/8-fermion term in the gradient expansion of $N = 2$ supersymmetric QCD.

August 1997

1. Introduction

Since their discovery, instantons have continued to play a central role in our understanding of non-perturbative effects in gauge theory. They are unique in providing non-perturbative effects which can nevertheless be calculated systematically in the semi-classical limit. Usually the main technical obstacle in such a calculation is to determine the correct quantum measure for integration over the collective coordinates of the instanton. In the case of a single instanton (topological charge $n = 1$), this measure was determined by 't Hooft in the classic paper [1]. In general, the complete measure is unknown for $n > 1$ although, as we will discuss below, an important part of the answer has been determined for the case $n = 2$. In this paper we will propose explicit expressions for the n-instanton measure for the gauge group $SU(2)$, both in theories with $N = 1$ supersymmetry (SUSY) (see Eqs. (2.23) and (2.54) below) and also in theories with $N = 2$ SUSY (see Eqs. (3.19) and (3.27) below). As non-trivial tests of our proposals, we will show that they reproduce known results in the one- and two-instanton sectors.

Instanton effects are especially prominent in the $N = 2$ theories, where they provide the only corrections to the holomorphic prepotential $\mathcal{F}$ beyond one-loop in perturbation theory [2-4]. Our expression given below for the $N = 2$ multi-instanton measure, combined with earlier results from Ref. [5], yields a closed expression for the n-instanton contribution to $\mathcal{F}$ as a finite-dimensional integral over bosonic and fermionic collective coordinates. Related expressions will be given for the 4-derivative/8-fermion term in the gradient expansion of $N = 2$ supersymmetric QCD.

The relevant field configurations in the $N = 1$ and $N = 2$ theories discussed above are supersymmetric extensions of the general multi-instanton solutions constructed by Atiyah, Drinfeld, Hitchin and Manin (ADHM) [6]. The ADHM construction reduces the problem of solving the self-dual Yang-Mills equation to that of solving purely algebraic equations. Despite this major simplification, the resulting algebraic constraints are highly non-linear and cannot be solved explicitly for general n. In calculating physical quantities the basic aim is to integrate over the solution space of the ADHM constraints (modulo an internal $O(n)$ redundancy described below). It is often assumed that knowledge of an explicit solution of the constraints is a prerequisite for constructing the integration measure; this is the source of the widely-held belief that progress is impossible beyond $n = 2$.[1]

In this paper we will adopt an alternative approach to this problem: we will define the collective coordinate integral as an integral over a larger set of variables with the supersymmetrized ADHM constraints imposed by δ-functions under the integral sign. The

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[1] In fact a solution of the ADHM constraints for $n = 3$ was presented in [7], but the complexity of the resulting expressions is such that no subsequent progress has been made for this case.

problem then is shifted to that of determining the correct algebraic form which sits inside the integrand (including the Jacobians induced by the δ-functions). The key simplification provided by SUSY which makes this determination possible stems from the fact that the $N = 1$ and $N = 2$ algebras can be realized as transformations of the instanton moduli [8,5]; in its most symmetric form, the integration measure must then be invariant under these transformations. (This simplification is in addition to the well-known observation that in a supersymmetric theory, the small-fluctuations 't Hooft determinants cancel in a self-dual background between bosonic and fermionic excitations [9].) This invariance requirement, together with the other symmetries of the problem, turns out to be sufficient to fix the multi-instanton measure up to a multiplicative constant for each n. These constants are then determined by induction, up to a single overall normalization, by the clustering property of the measure in the dilute gas limit where the separation between instantons is much larger than their scale sizes. Finally, this single remaining normalization is fixed by comparison to 't Hooft's result in the one-instanton sector. As mentioned above, comparison to known expressions in the two-instanton sector [10,11] then yields an independent test of our proposals. It would be interesting to see if the collective coordinate measures given below can be deduced in an alternative way familiar from the study of monopoles, by constructing the (supersymmetrized) volume form from the hyper-Kähler metric 2-form on the ADHM space.

The paper is organized as follows. Section 2 is devoted to the case of $N = 1$ supersymmetric $SU(2)$ gauge theory. After a brief review of our ADHM and SUSY conventions, we write down our Ansatz for the collective coordinate integration measure, which is invariant not only under SUSY, but also under the internal $O(n)$ symmetry mentioned above. Much of this section is devoted to analyzing the clustering requirement, which fixes the overall normalization constants of the n-instanton measure by induction in n. Clustering together with zero-mode counting turns out to be a highly restrictive requirement, ruling out virtually all other conceivable SUSY- and $O(n)$-invariant forms for the measure. At the end of Sec. 2 we show that for $n = 2$ our postulated measure is equivalent to the known first-principles 2-instanton measure constructed in [10,11], in which neither the SUSY nor the $O(2)$ invariance is manifest (instead these symmetries are "gauge-fixed"). In Sec. 3 we repeat all these steps for the case of $N = 2$ supersymmetric $SU(2)$ gauge theory. Section 4 briefly discusses the incorporation of matter supermultiplets, which poses no problems.

In Sec. 5 we revisit the Seiberg-Witten prepotential $\mathcal{F}$, and give the explicit collective coordinate integral formula for $\mathcal{F}_n$, the n-instanton contribution to $\mathcal{F}$, valid for any number of flavors N_F. This is tantamount to an all-instanton-orders solution in quadratures of the Seiberg-Witten models, without appeal to electric-magnetic duality. (For analogous multi-instanton solutions of certain 2-dimensional and 3-dimensional models, see for instance

Refs. [12] and [13], respectively.) While such a solution may be of purely academic interest for $N_F \leq 3$ in light of the exact results of [3,4], for the conformal case $N_F = 4$ it gives something new: the all-instanton-orders relation between the microscopic and effective coupling constants, τ_{micro} and τ_{eff}, which are known to be inequivalent [5,14]. So far as we currently understand, $SL(2, \mathbb{Z})$ duality by itself cannot give this relation, as it operates only at the level of τ_{eff}. Using the results of [15], we give similar expressions for the pure-holomorphic and pure-antiholomorphic contributions to the 4-derivative/8-fermion term in the gradient expansion along the Coulomb branch of $N = 2$ supersymmetric QCD. Section 5 concludes with comments about how one actually performs these collective coordinate integrations, although explicit calculational progress along these lines is deferred to future work.

2. The $N = 1$ supersymmetric collective coordinate integration measure

2.1. ADHM and SUSY review

The basic object in the ADHM construction [6] of self-dual $SU(2)$ gauge fields of topological number n is an $(n+1) \times n$ quaternion-valued matrix $\Delta_{\lambda l}(x)$, which is a linear function of the space-time variable x:[2]

$$\Delta_{\lambda l} = a_{\lambda l} + b_{\lambda l} x , \quad 0 \leq \lambda \leq n , \quad 1 \leq l \leq n . \tag{2.1}$$

The gauge field $v_m(x)$ is then given by (displaying color indices)

$$v_m{}^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{U}_\lambda^{\dot{\alpha}\alpha} \partial_m U_{\lambda \alpha \dot{\beta}} , \tag{2.2}$$

where the quaternion-valued vector U_λ lives in the $\perp$ space of Δ:

$$\bar{\Delta}_{l\lambda} U_\lambda = \bar{U}_\lambda \Delta_{\lambda l} = 0 , \tag{2.3a}$$

$$\bar{U}_\lambda U_\lambda = 1 . \tag{2.3b}$$

It is easy to show that self-duality of the field strength v_{mn} is equivalent to the quaternionic condition

$$\bar{\Delta}_{k\lambda}^{\dot{\beta}\beta} \Delta_{\lambda l \beta \dot{\alpha}} = (f^{-1})_{kl} \delta^{\dot{\beta}}{}_{\dot{\alpha}} \tag{2.4}$$

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[2] We use quaternionic notation $x = x_{\alpha\dot{\alpha}} = x_n \sigma^n_{\alpha\dot{\alpha}}$, $\bar{a} = \bar{a}^{\dot{\alpha}\alpha} = \bar{a}^n \bar{\sigma}_n^{\dot{\alpha}\alpha}$, $b = b_\alpha{}^\beta$, etc., where σ^n and $\bar{\sigma}^n$ are the spin matrices of Wess and Bagger [16]. See Ref. [11] for a self-contained introduction to the ADHM construction including a full account of our ADHM and SUSY conventions. We also set the coupling constant $g = 1$ throughout, except, for clarity, in the Yang-Mills instanton action $8\pi^2 n/g^2$.

for some scalar-valued $n \times n$ matrix f; Taylor expanding in x then gives

$$\bar{a}a = (\bar{a}a)^T \propto \delta^{\dot{\beta}}{}_{\dot{\alpha}} , \qquad (2.5a)$$

$$\bar{b}a = (\bar{b}a)^T , \qquad (2.5b)$$

$$\bar{b}b = (\bar{b}b)^T \propto \delta_{\alpha}{}^{\beta} . \qquad (2.5c)$$

Here the T stands for transpose in the ADHM indices $(\lambda, l, \text{etc.})$ only, whereas an overbar indicates conjugation in both the ADHM and the quaternionic spaces.

Following Refs. [17,7,11], we will work in a representation in which b assumes a simple canonical form, namely

$$a_{\alpha\dot{\alpha}} = \begin{pmatrix} w_{1\alpha\dot{\alpha}} & \cdots & w_{n\alpha\dot{\alpha}} \\ & a'_{\alpha\dot{\alpha}} & \end{pmatrix} , \qquad b_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & \cdots & 0 \\ \delta_{\alpha}{}^{\beta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{\alpha}{}^{\beta} \end{pmatrix} \qquad (2.6)$$

Thanks to this simple form for b, the constraint $(2.5c)$ is now automatically satisfied, while $(2.5b)$ reduces to the symmetry condition on the $n \times n$ submatrix a':

$$a' = a'^T \qquad (2.7)$$

Note that there is an $O(n)$ group of transformations on $\Delta(x)$ which preserves this canonical form for b as well as the ADHM conditions (2.5), but acts nontrivially on a:

$$\Delta \rightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R^T & \\ 0 & & & \end{pmatrix} \cdot \Delta \cdot R , \quad f \rightarrow R^T \cdot f \cdot R , \quad U_0 \rightarrow U_0 , \quad U_l \rightarrow R_{kl} U_k \quad (2.8)$$

Here R is an $O(n)$ matrix whose elements are independent of x and act by scalar multiplication on the quaternions. From (2.2) one sees that these transformations do not affect the gauge field v_m. Hence, the physical moduli space, $\mathrm{M}_{\mathrm{phys}}$, of gauge-inequivalent self-dual gauge configurations is the quotient of the space M of all solutions of the constraints (2.5) which have the canonical form (2.6), by the symmetry group $O(n)$:

$$\mathrm{M}_{\mathrm{phys}} = \frac{\mathrm{M}}{O(n)} \qquad (2.9)$$

Let us count the number of physical degrees of freedom represented by the collective coordinate matrix a. Since the elements of a are quaternions, it contains $4(n + \frac{1}{2}n(n+1))$ real scalar degrees of freedom thanks to (2.7). The ADHM constraint $(2.5a)$ then imposes

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$\frac{3}{2}n(n-1)$ conditions on the upper-triangular traceless quaternionic elements of $\bar{a}a$, while the modding out by the $O(n)$ action (2.8) subtracts an additional $\frac{1}{2}n(n-1)$ degrees of freedom. This leaves

$$4(n + \tfrac{1}{2}n(n+1)) \ - \ \tfrac{3}{2}n(n-1) \ - \ \tfrac{1}{2}n(n-1) \ = \ 8n \tag{2.10}$$

physical degrees of freedom for the multi-instanton with topological number n. This is the correct number: in the limit of n widely separated (i.e., distinguishable) instantons it properly accounts for $4n$ positions, $3n$ iso-orientations, and n instanton scale sizes.

In an $N = 1$ supersymmetric theory the gauge field v is accompanied by a gaugino λ. By the index theorem, the zero modes of λ should comprise $4n$ Grassmann (i.e., anticommuting) degrees of freedom. These adjoint fermion zero modes have the form [17]

$$(\lambda_\alpha)^{\dot\beta}{}_{\dot\gamma} \ = \ \bar{U}^{\dot\beta\gamma}\mathcal{M}_\gamma f\bar{b}\, U_{\alpha\dot\gamma} \ - \ \bar{U}^{\dot\beta}{}_\alpha \, bf\mathcal{M}^{\gamma T} U_{\gamma\dot\gamma} \ . \tag{2.11}$$

We suppress ADHM matrix indices (λ, l, etc.) but exhibit color (dotted) and Weyl (undotted) indices for clarity. Here $\mathcal{M}$ is an x-independent Weyl-spinor-valued $(n+1) \times n$ matrix:

$$\mathcal{M}^\gamma \ = \ \begin{pmatrix} \mu_1^\gamma & \cdots & \mu_n^\gamma \\ & \mathcal{M}'^\gamma & \end{pmatrix} \tag{2.12}$$

The 2-component Dirac equation in the background of the ADHM multi-instanton (2.2) is equivalent to the following linear constraints on $\mathcal{M}_\gamma$ [17]:

$$\bar{a}^{\dot\alpha\gamma}\mathcal{M}_\gamma \ = \ -\mathcal{M}^{\gamma T} a_\gamma{}^{\dot\alpha} \ , \tag{2.13a}$$

$$\mathcal{M}_\gamma'^T \ = \ \mathcal{M}_\gamma' \ . \tag{2.13b}$$

This leaves

$$2\big(n + \tfrac{1}{2}n(n+1)\big) \ - \ n(n-1) \ = \ 4n \tag{2.14}$$

degrees of freedom in $\mathcal{M}$, as is needed. Under $O(n)$, $\mathcal{M}$ transforms just like $\Delta(x)$, Eq. (2.8).

Next we review the supersymmetric properties of the collective coordinate matrices a and $\mathcal{M}$ [5]. As the relevant field configurations v_m and λ_α obey equations of motion which are manifestly supersymmetric, any non-vanishing action of the supersymmetry generators on a particular solution necessarily yields another solution. It follows that the "active" supersymmetry transformations of the fields must be equivalent (up to a gauge transformation) to certain "passive" transformations of the $8n$ independent bosonic and $4n$ independent fermionic collective coordinates which parametrize the superinstanton

solution. As originally noted in [8], physically relevant quantities such as the saddle-point action of the superinstanton must be constructed out of supersymmetric invariant combinations of the collective coordinates.

An especially attractive feature of the ADHM construction is that the supersymmetry algebra can actually be realized directly as transformations of the highly over-complete (order n^2, rather than order n) set of collective coordinates a and $\mathcal{M}$. Under an infinitesimal supersymmetry transformation $\xi Q + \bar{\xi}\bar{Q}$, these transform as [5]:[3]

$$\delta a_{\alpha\dot\alpha} = \bar{\xi}_{\dot\alpha}\mathcal{M}_\alpha \tag{2.15a}$$

$$\delta \mathcal{M}_\gamma = -4ib\xi_\gamma \tag{2.15b}$$

This algebra allows us to promote a to a space-time-constant "superfield" $a(\bar\theta)$ in an obvious way:[4]

$$a_{\alpha\dot\alpha} \;\rightarrow\; a_{\alpha\dot\alpha}(\bar\theta) \;=\; e^{\bar\theta\bar Q} \times a_{\alpha\dot\alpha} \;=\; a_{\alpha\dot\alpha} + \bar\theta_{\dot\alpha}\mathcal{M}_\alpha \ . \tag{2.16}$$

In superfield language the bosonic and fermionic constraints (2.5a) and (2.13a) assemble naturally into a supermultiplet of constraints, namely [5]:

$$\bar{a}(\bar\theta)a(\bar\theta) \;=\; \big(\bar{a}(\bar\theta)a(\bar\theta)\big)^T \;\propto\; \delta^{\dot\beta}{}_{\dot\alpha} \ . \tag{2.17}$$

The scalar piece of (2.17) gives (2.5a) and the $\mathcal{O}(\bar\theta)$ piece gives (2.13a), while the $\mathcal{O}(\bar\theta^2)$ piece is "auxiliary" as it is satisfied automatically.

2.2. Ansatz for the measure

In this section we will discuss the $N = 1$ superinstanton measure $d\mu^{(n)}_{\rm phys}$, for arbitrary topological number n. As the small-fluctuations determinants in a self-dual background cancel between the bosonic and fermionic sectors in a supersymmetric theory [9], the relevant measure is the one inherited from the Feynman path integral on changing variables from the fields to the collective coordinates which parametrize the instanton moduli space $\mathrm{M}_{\rm phys}$. In principle the super-Jacobian for this change of variables can be calculated by evaluating the normalization matrices of the appropriate bosonic and fermionic zero-modes. In practice, this involves solving the ADHM constraints (2.5) and can only be accomplished for $n \leq 2$.

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[3] In the 1-instanton sector, these SUSY transformations are equivalent, up to an $SU(2)$ gauge transformation, to those of Novikov, Shifman, Vainshtein and Zakharov [8].

[4] Here we ignore the action of the Q_α. These generators correspond to supersymmetries which are broken by the self-dual gauge-field configuration and they act on the moduli in a trivial way.

As discussed in Section 1, we will pursue an alternative approach to the problem of determining the correct measure. The first step is to formally undo the $O(n)$ quotient described in Eq. (2.9) and define an unidentified measure, $d\mu^{(n)}$, for integration over the larger moduli space M:

$$\int_{M_{\text{phys}}} d\mu^{(n)}_{\text{phys}} \equiv \frac{1}{\text{Vol}(O(n))} \int_M d\mu^{(n)} \tag{2.18}$$

The correctly normalized volumes for the $O(n)$ groups follow from

$$O(n) = \frac{O(n)}{O(n-1)} \times \frac{O(n-1)}{O(n-2)} \times \cdots \times \frac{O(2)}{O(1)} \times O(1) \tag{2.19}$$

and

$$\frac{O(n)}{O(n-1)} = \frac{SO(n)}{SO(n-1)} = S^{n-1} \tag{2.20}$$

where S^{n-1} is the $(n-1)$-sphere. Consequently the group volumes are fixed by the recursion relation

$$\text{Vol}(O(n)) = 2 \cdot \text{Vol}(SO(n)) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \text{Vol}(O(n-1)) \tag{2.21}$$

together with the initial condition

$$\text{Vol}(O(1)) = 2 . \tag{2.22}$$

We will now seek a measure, $d\mu^{(n)}$, with the following five properties:

(i) $O(n)$ invariance;

(ii) supersymmetry invariance;

(iii) a net of $8n$ unconstrained bosonic integrations and also $4n$ unsaturated Grassmann integrations over the parameters of the adjoint zero modes ($8n$ in the $N = 2$ case);

(iv) cluster decomposition in the dilute-gas limit of large space-time separation between instantons;

(v) agreement with known formulae in the 1-instanton sector.

Taken together, these are very restrictive requirements and we claim that they uniquely determine the measure $d\mu^{(n)}$. In particular, we conclude that the following Ansatz is the unique solution to conditions (i), (ii) and (iii):

$$\begin{aligned}
\int d\mu^{(n)}_{\text{phys}} &\equiv \frac{1}{\text{Vol}(O(n))} \int d\mu^{(n)} \\
&= \frac{C_n}{\text{Vol}(O(n))} \int \prod_{i=1}^{n} d^4 w_i d^2 \mu_i \prod_{(ij)_n} d^4 a'_{ij} d^2 \mathcal{M}'_{ij} \\
&\quad \times \prod_{\langle ij \rangle_n} \prod_{c=1,2,3} \delta\left(\tfrac{1}{4}\text{tr}_2 \, \tau^c [(\bar{a}a)_{i,j} - (\bar{a}a)_{j,i}]\right) \delta^2\left((\bar{a}\mathcal{M})_{i,j} - (\bar{a}\mathcal{M})_{j,i}\right) .
\end{aligned} \tag{2.23}$$

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where the collective coordinates w_i, μ_i, a'_{ij} and $\mathcal{M}'_{ij}$ were defined in Eqs. (2.6) and (2.12). The notation $(ij)_n$ and $\langle ij \rangle_n$, which we will use for symmetric and antisymmetric $n \times n$ matrices respectively, stands for the ordered pairs (i, j) restricted as follows:

$$(ij)_n : \qquad 1 \leq i \leq j \leq n \tag{2.24a}$$

$$\langle ij \rangle_n : \qquad 1 \leq i < j \leq n \tag{2.24b}$$

Further, we will show below that the overall numerical constants C_n are determined by condition (iv) up to a single normalization which is, in turn, fixed by condition (v).

The two δ-functions in (2.23) implement the constraints (2.5a) and (2.13a), respectively. In the case of the 1-instanton measure these constraints disappear, and one simply has

$$\int d\mu_{\mathrm{phys}}^{(1)} \;\equiv\; \tfrac{1}{2} \int d\mu^{(1)} \;=\; \tfrac{1}{2} C_1 \int d^4w \, d^2\mu \, d^4a' \, d^2\mathcal{M}' \;. \tag{2.25}$$

This is precisely 't Hooft's 1-instanton measure [1], rewritten in the quaternionic notation of ADHM. In particular the position, size, and $SU(2)$ iso-orientation of the instanton are given, respectively, by a', $|w|$, and $w/|w|$, with the fermionic quantities in (2.25) denoting their respective superpartners. Also $\tfrac{1}{2} C_1$ is 't Hooft's scheme-dependent 1-instanton factor.

In the remainder of Sec. 2 we verify that the measure (2.23) satisfies the above mentioned properties and then perform a highly non-trivial check on our Ansatz by comparing it to the known results in the two-instanton sector.

2.3. $O(n)$ invariance, SUSY invariance, and dimensional power counting

The $O(n)$ invariance of this measure is obvious by inspection. As for supersymmetry invariance under (2.15), this too is obvious for ξQ, while for $\bar{\xi}\bar{Q}$ the reasoning is as follows: the argument of the second δ-function in (2.23) (which implements (2.13a)) is invariant, while that of the first δ-function (which implements (2.5a)) transforms into itself plus an admixture of the second under (2.15a), so that the product of δ-functions is an invariant. The underlying reason for this is, of course, that the constraints (2.5) and (2.13) form a supermultiplet, Eq. (2.17).

Next we verify the counting requirement (iii). Recall that by the rules of Grassmann integration,[5] a δ-function of a Grassmann-valued argument may simply be replaced by the argument itself. So, of the $n^2 + 3n$ fermionic μ_i and $\mathcal{M}'_{ij}$ modes in (2.23), $n(n-1)$ of them are saturated by the second δ-function, leaving $4n$ unbroken gaugino zero modes as

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[5] We remind the reader of the complete "Gradshteyn and Ryzhik" for Grassmann integration: $\int d\chi = 0$ and $\int d\chi\, \chi = 1$.

required. As a further test of our Ansatz one may check that (2.23) has the correct bosonic dimensionality. The bosonic sector scales like

$$[a]^{(2n^2+6n)-(3n^2-3n)+(n^2-n)} \sim [a]^{8n} , \tag{2.26}$$

the three terms in the exponent coming, respectively, from the w_i and a'_{ij} integration variables, from the first δ-function, and from the second δ-function.

2.4. Cluster decomposition

Next we examine the clustering property (iv), which turns out to be the least obvious, and most stringent, of the requirements. We will analyze the limit in which one of the instanton position moduli is far away from all the others, and demand that the measure (2.23) factor approximately into a product of a 1-instanton and an $(n-1)$-instanton measure. Recall that in the limit of large separation, the space-time positions of the n individual instantons making up the topological-number-n configuration may simply be identified with the n diagonal elements a'_{ii} [7]. A convenient clustering limit is then

$$|a'_{nn}| \rightarrow \infty \tag{2.27}$$

with all other elements of a remaining "order unity" (or smaller, if dictated by the ADHM constraints). Of course (2.27) is not an $O(n)$-invariant statement. We can rewrite (2.27) as the statement that a rank-one submatrix h of a', defined as

$$h = q \cdot V \cdot V^T , \tag{2.28}$$

where q is a quaternion and V is a unit-normalized n-vector in $\mathbb{R}^n$, becomes large:

$$|q| \rightarrow \infty . \tag{2.29}$$

Equation (2.27) then corresponds to the choice

$$V = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} , \qquad q = a'_{nn} . \tag{2.30}$$

Note that an $O(n-1)$ subgroup of $O(n)$ leaves the choice (2.28) and (2.30) invariant; (2.28) is only acted on by the coset $O(n)/O(n-1)$ which sweeps the vector V through the $(n-1)$-sphere S^{n-1}. To make this more precise, we parametrize $SO(n) \subset O(n)$ by the set of $n \times n$ generators t_{ij}, $1 \leq i < j \leq n$, defined by their matrix elements

$$\left(t_{ij}\right)_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} . \tag{2.31}$$

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Every $g \in SO(n)$ can be written as [18]

$$g = \prod_{j=1}^{n} \exp\left(-\sum_{i=1}^{j-1} \alpha_{ij}\, t_{ij}\right) . \tag{2.32}$$

In these coordinates the properly normalized Haar measure for $SO(n)$ takes the form of a nested product of cosets, as per Eq. (2.19):

$$dg = \prod_{j=1}^{n} dg_j , \qquad dg_j = \left(\prod_{i=1}^{j-1} d\alpha_{ij}\right) \frac{1}{\sqrt{1 - \sum_{i=1}^{j-1} \alpha_{ij}^2}} \tag{2.33}$$

while the group volumes (2.21) follow from

$$\mathrm{Vol}\big(SO(n)\big) \equiv \int_{SO(n)} dg = \prod_{j=1}^{n} 2 \int_{D(j-1)} dg_j \tag{2.34}$$

Here the domain of integration $D(j-1)$ is the unit $(j-1)$-ball, and the factors of 2 count the two hemispheres of each coset. For infinitesimal α_{ij}, g acts on the submatrix h defined by Eqs. (2.28) and (2.30) as follows:

$$
a'_{nn} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0, \cdots, 0, 1 \end{pmatrix} \longrightarrow a'_{nn}\, g^T \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0, \cdots, 0, 1 \end{pmatrix} \cdot g
$$

$$
= a'_{nn} \cdot \begin{pmatrix} 0 & \cdots & 0 & \alpha_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{n-1,n} \\ \alpha_{1n} & \cdots & \alpha_{n-1,n} & 1 \end{pmatrix} + \mathcal{O}(\alpha_{ij}^2) . \tag{2.35}
$$

Equation (2.35) defines the infinitesimal action of the coset $O(n)/O(n-1) \cong SO(n)/SO(n-1)$ on h.

We can now state precisely what is meant by the clustering condition in the limit $|a'_{nn}| \to \infty$. It is important to stress that clustering is a property of the unidentified measure $d\mu^{(n)}$, rather than of the physical measure $d\mu^{(n)}_{\mathrm{phys}}$ in which points on the $O(n)$ orbit are identified. This distinction is emphasized in Refs. [19,10] (albeit in a formalism in which it is a finite subgroup of $O(n)$ rather than the full $O(n)$ that is being modded out). Furthermore, for the unidentified measure, one cannot simply demand $d\mu^{(n)} \to d\mu^{(n-1)} \times d\mu^{(1)}$ since the number of bosonic differentials on the left-hand side exceeds that of the right-hand side by $\dim(O(n)) - \dim(O(n-1)) = n - 1$. Instead, the proper clustering condition reads

$$d\mu^{(n)} \xrightarrow{|a'_{nn}| \to \infty} d\mu^{(n-1)} \times d\mu^{(1)} \times dS^{n-1} \tag{2.36}$$

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Here $d\mu^{(n-1)}$ is built from the variables $\{w_j, \mu_j, a'_{ij}, \mathcal{M}'_{ij}\}$ with $1 \leq i \leq j \leq n-1$; $d\mu^{(1)}$ is built from $\{w_n, \mu_n, a'_{nn}, \mathcal{M}'_{nn}\}$; and

$$dS^{n-1} \;=\; \prod_{i=1}^{n-1} d\alpha_{in} \tag{2.37}$$

in the notation of Eqs. (2.32)-(2.35). Note from Eq. (2.33) that Eq. (2.37) is only correct for V^T in an infinitesimal neighborhood of $(0, \cdots, 0, 1)$, i.e. for infinitesimal α_{in}; this is all that is actually needed for present purposes.

The calculation below proceeds as follows. In the fermionic sector, the $n-1$ extra δ-functions on the left-hand side of (2.36), namely $\prod_{i=1}^{n-1} \delta^2\big((\bar{a}\mathcal{M})_{i,n} - (\bar{a}\mathcal{M})_{n,i}\big)$, which have no counterparts on the right-hand side, are killed upon integration of the $n-1$ extra differentials $\prod_{i=1}^{n-1} d^2\mathcal{M}'_{in}$. The bosonic sector is more complicated: the extra δ-functions $\prod_{i=1}^{n-1} \prod_{c=1,2,3} \delta\big(\frac{1}{4}\mathrm{tr}_2\,\tau^c[(\bar{a}a)_{i,n} - (\bar{a}a)_{n,i}]\big)$ are killed by three-quarters of the extra bosonic differentials $\prod_{i=1}^{n-1} \int d^4 a'_{in}$; the remaining one-quarter assemble to form the differential dS^{n-1} on the right-hand side of Eq. (2.36). In this way we shall verify the condition (2.36), and in so doing, obtain a formula for the overall factors C_n. Here are the details:

Let us return to the measure $d\mu^{(n)}$, Eq. (2.23), and perform, first, the subset of Grassmann integrations

$$\int \prod_{i=1}^{n-1} d^2\mathcal{M}'_{in}\, \delta^2\big((\bar{a}\mathcal{M})_{i,n} - (\bar{a}\mathcal{M})_{n,i}\big) \times \cdots \tag{2.38}$$

The dots represent the remaining δ-functions in the integrand of (2.23). We expand

$$(\bar{a}\mathcal{M})_{i,n} - (\bar{a}\mathcal{M})_{n,i} \;=\; -\bar{a}'_{nn}\mathcal{M}'_{in} + \sum_{k=1}^{n-1} \bar{a}'_{ik}\mathcal{M}'_{kn} + \cdots, \tag{2.39}$$

the dots representing modes other than the $\mathcal{M}'_{kn}$ modes. Hence the integration (2.38) kills the δ-functions and produces the Jacobian

$$J_{\mathcal{M}} \;=\; |a'_{nn}|^{2(n-1)} \;+\; \cdots, \tag{2.40}$$

neglecting subleading powers of $|a'_{nn}|$. Next, one carries out the analogous subset of bosonic integrations

$$\int \prod_{i=1}^{n-1} d^4 a'_{in} \prod_{c=1,2,3} \delta\big(\tfrac{1}{4}\mathrm{tr}_2\,\tau^c[(\bar{a}a)_{i,n} - (\bar{a}a)_{n,i}]\big) \times \cdots \tag{2.41}$$

in the following manner. First one changes quaternionic variables to

$$a'_{in\,\alpha\dot\alpha} \;=\; a'_{nn\,\alpha\dot\beta}\,\hat{a}^{\dot\beta}_{in\,\dot\alpha}\,, \qquad 1 \leq i \leq n-1\,, \tag{2.42}$$

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where $\hat{a}_{in}$ in turn is divided into self-bar (SB) and anti-self-bar (ASB) pieces, thus:

$$\hat{a}_{in} = \hat{a}_{in}^{\rm SB} + \hat{a}_{in}^{\rm ASB} , \quad \bar{\hat{a}}_{in}^{\rm SB} = \hat{a}_{in}^{\rm SB} , \quad \bar{\hat{a}}_{in}^{\rm ASB} = -\hat{a}_{in}^{\rm ASB} . \tag{2.43}$$

Note that $\hat{a}_{in}^{\rm SB}$ is a scalar: $(\hat{a}_{in}^{\rm SB})^{\dot{\beta}}{}_{\dot{\alpha}} \propto \delta^{\dot{\beta}}{}_{\dot{\alpha}}$. In these variables the integration reads

$$\int \prod_{i=1}^{n-1} d^4 a'_{in} = |a'_{nn}|^{4(n-1)} \int \prod_{i=1}^{n-1} d^3 \hat{a}_{in}^{\rm ASB} \, d\hat{a}_{in}^{\rm SB} . \tag{2.44}$$

and the argument of the δ-function in (2.41) becomes

$$(\bar{a}a)_{i,n} - (\bar{a}a)_{n,i}$$
$$= -2\hat{a}_{in}^{\rm ASB} |a'_{nn}|^2 + \sum_{k=1}^{n-1} \left(\bar{a}'_{ik} a'_{nn} (\hat{a}_{kn}^{\rm SB} + \hat{a}_{kn}^{\rm ASB}) - (\hat{a}_{kn}^{\rm SB} - \hat{a}_{kn}^{\rm ASB}) \bar{a}'_{nn} a'_{ki} \right) + \cdots \tag{2.45}$$

where the dots represent terms that do not depend on the $\hat{a}_{kn}$. Performing the $d^3 \hat{a}_{in}^{\rm ASB}$ integration kills the δ-functions in (2.41) and produces the Jacobian

$$J_a = \left(\frac{1}{|a'_{nn}|^2} \right)^{3(n-1)} + \cdots, \tag{2.46}$$

again neglecting terms subleading in $|a'_{nn}|$. Note that the powers of $|a'_{nn}|$ precisely cancel among Eqs. (2.40), (2.44) and (2.46).

It remains to carry out the $n-1$ integrations over the $\hat{a}_{in}^{\rm SB}$. But these integration variables, viewed as infinitesimals, are precisely the generators (2.35) of $O(n)/O(n-1)$ that sweep the vector V in (2.28) through S^{n-1}:

$$\hat{a}_{in}^{\rm SB} = \alpha_{in} , \qquad \prod_{i=1}^{n-1} d\hat{a}_{in}^{\rm SB} = dS^{n-1} . \tag{2.47}$$

Gathering the results to this point, we have shown that in the limit $|a'_{nn}| \to \infty$,

$$\int d\mu^{(n)} \longrightarrow C_n \int dS^{n-1} \times \int dw_n d\mu_n d^4 a'_{nn} d^2 \mathcal{M}'_{nn}$$

$$\times \int \prod_{i=1}^{n-1} d^4 w_i d^2 \mu_i \prod_{(ij)_{n-1}} d^4 a'_{ij} d^2 \mathcal{M}'_{ij} \tag{2.48}$$

$$\times \prod_{\langle ij \rangle_{n-1}} \prod_{c=1,2,3} \delta\left(\tfrac{1}{4} \mathrm{tr}_2 \, \tau^c [(\bar{a}a)_{i,j} - (\bar{a}a)_{j,i}] \right) \delta^2 \left((\bar{a}\mathcal{M})_{i,j} - (\bar{a}\mathcal{M})_{j,i} \right) .$$

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The second integral in the first line is proportional to the 1-instanton measure $d\mu^{(1)}$ as anticipated. Let us define $\tilde{a}$ and $\tilde{\mathcal{M}}$ to be the truncated versions of a and $\mathcal{M}$, with the last row and column lopped off:

$$\tilde{a} = \begin{pmatrix} w_1 & \cdots & w_{n-1} \\ a'_{11} & \cdots & a'_{1,n-1} \\ \vdots & \ddots & \vdots \\ a'_{1,n-1} & \cdots & a'_{n-1,n-1} \end{pmatrix}, \qquad \tilde{\mathcal{M}} = \begin{pmatrix} \mu_1 & \cdots & \mu_{n-1} \\ \mathcal{M}'_{11} & \cdots & \mathcal{M}'_{1,n-1} \\ \vdots & \ddots & \vdots \\ \mathcal{M}'_{1,n-1} & \cdots & \mathcal{M}'_{n-1,n-1} \end{pmatrix} \tag{2.49}$$

We still need to verify that the last two lines of (2.48) are proportional to the $(n-1)$-instanton measure $d\mu^{(n-1)}$, meaning that a and $\mathcal{M}$ can be approximately replaced by $\tilde{a}$ and $\tilde{\mathcal{M}}$ inside the δ-functions. The error in making these approximations is:

$$[(\bar{a}a)_{i,j} - (\bar{a}a)_{j,i}] - [(\bar{\tilde{a}}\tilde{a})_{i,j} - (\bar{\tilde{a}}\tilde{a})_{j,i}] = |a'_{nn}|^2 \left[\hat{a}^{\text{ASB}}_{jn}, \hat{a}^{\text{ASB}}_{in} \right] \tag{2.50}$$

and

$$[(\bar{a}\mathcal{M})_{i,j} - (\bar{a}\mathcal{M})_{j,i}] - [(\bar{\tilde{a}}\tilde{\mathcal{M}})_{i,j} - (\bar{\tilde{a}}\tilde{\mathcal{M}})_{j,i}] = \hat{a}^{\text{ASB}}_{jn}\bar{a}'_{nn}\mathcal{M}'_{in} - \hat{a}^{\text{ASB}}_{in}\bar{a}'_{nn}\mathcal{M}'_{jn}, \tag{2.51}$$

where we neglect the $\hat{a}^{\text{SB}}_{in}$ as we are focusing on an infinitesimal neighborhood of V. From Eqs. (2.45) and (2.39) one learns that

$$\hat{a}^{\text{ASB}}_{in} \sim |a'_{nn}|^{-2}, \qquad \mathcal{M}'_{in} \sim |a'_{nn}|^{-1} \tag{2.52}$$

so that the right-hand sides of (2.50) and (2.51) indeed vanish like $|a'_{nn}|^{-2}$ in the clustering limit, as desired. The clustering condition (2.36), applied to Eq. (2.48), then collapses to the simple numerical recursion

$$C_n = C_{n-1} \cdot C_1 \tag{2.53}$$

or equivalently

$$C_n = (C_1)^n. \tag{2.54}$$

This formula completes the specification of the $N = 1$ supersymmetric instanton measure (2.23).

2.5. Agreement in the 2-instanton sector

Finally we verify that our Ansatz (2.23) and (2.54) for the $N = 1$ supersymmetric measure $d\mu^{(n)}_{\text{phys}}$ agrees with previously known results for topological number $n = 1$ [1] and $n = 2$ [10,11]. The case $n = 1$ was already discussed in Eq. (2.25) *ff.* The case $n = 2$

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is more interesting. Following [10,11], $d\mu_{\text{phys}}^{(2)}$ is known from first principles to have the following form:

$$\int d\mu_{\text{phys}}^{(2)} = \frac{1}{\mathcal{S}_2} \int d^4 w_1 d^4 w_2 d^4 a'_{11} d^4 a'_{22} d^2 \mu_1 d^2 \mu_2 d^2 \mathcal{M}'_{11} d^2 \mathcal{M}'_{22} \left(J_{\text{bose}}/J_{\text{fermi}} \right)^{1/2} \quad (2.55)$$

The definition of the zero-mode Jacobians J_{bose} and J_{fermi} and the discrete group-theoretic $\mathcal{S}_2$ will be reviewed below. Note that this particular form for the measure breaks $O(2)$ invariance as it involves integration over only the diagonal elements of a' and $\mathcal{M}'$, the off-diagonal elements having been eliminated by the explicit resolution of all the constraints (which is elementary to do only for $n = 2$). At the same time it breaks SUSY invariance, since (as reviewed shortly) J_{bose} and J_{fermi} are purely bosonic expressions with no fermion bilinear parts. Our goal is to demonstrate that this measure is nevertheless equivalent to the $O(2)$- and SUSY-invariant form (2.23). In fact, the known expression (2.55) is precisely an "$O(2)$-gauge-fixed" version of (2.23). (We put this in quotes because $O(2)$ invariance is purely internal to the ADHM construction and has nothing to do with the usual choice of space-time gauge.) As shown below, to recover an $O(2)$-invariant form we will integrate (2.55) over $O(2)$ orbits; pleasingly, SUSY invariance is recovered simultaneously.

The quantities J_{fermi} and J_{bose} that enter (2.55) were obtained in Refs. [11] and [10], respectively. Up to a multiplicative constant, J_{fermi} has the form[6]

$$J_{\text{fermi}}^{1/2} \propto \frac{H}{|a_3|^2} \quad (2.56)$$

where

$$H = |w_1|^2 + |w_2|^2 + 4|a'_{12}|^2 + 4|a_3|^2 \quad (2.57)$$

and a_3 (likewise a_0, needed below) is shorthand for the linear combination

$$a_3 = \tfrac{1}{2}(a'_{11} - a'_{22}), \quad a_0 = \tfrac{1}{2}(a'_{11} + a'_{22}) . \quad (2.58)$$

The construction of J_{bose} is considerably more intricate, due to the nonlinearity of the bosonic constraint (2.5a) and the need to enforce the background gauge condition, and is the principal achievement of [10]. For $n = 2$ the general solution to (2.5a) is easily obtained, and reads:

$$a'_{12} = \frac{1}{4|a_3|^2} a_3(\bar{w}_2 w_1 - \bar{w}_1 w_2 + \Sigma) . \quad (2.59)$$

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[6] This expression is the square-root of the formula in [11], as the $N = 1$ theory contains half as many adjoint fermion zero modes as the $N = 2$ theory.

Here $\Sigma^{\dot\alpha}{}_{\dot\beta}$ is an arbitrary scalar-valued function of $\{a_3, a_0, w_1, w_2\}$ (which as per (2.55) we select to be our set of $8n = 16$ independent bosonic variables):

$$\Sigma^{\dot\alpha}{}_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}\,\Sigma(a_3, a_0, w_1, w_2) \; . \tag{2.60}$$

From (2.59) we can solve instead for Σ :

$$\Sigma = 2(\bar{a}_3 a'_{12} + \bar{a}'_{12} a_3) \; . \tag{2.61}$$

In these coordinates, Osborn's result for the bosonic Jacobian reads [10]

$$J_{\text{bose}}^{1/2} \propto \frac{H}{|a_3|^4}\left|\,|a_3|^2 - |a'_{12}|^2 - \tfrac{1}{8}\frac{d\Sigma^\phi}{d\phi}\Big|_{\phi=0}\right| \; , \tag{2.62}$$

so that

$$\left(J_{\text{bose}}/J_{\text{fermi}}\right)^{1/2} = C_2\,\frac{\left|\,|a_3|^2 - |a'_{12}|^2 - \tfrac{1}{8}\frac{d\Sigma^\phi}{d\phi}\big|_{\phi=0}\right|}{|a_3|^2} \; . \tag{2.63}$$

The quantity Σ^ϕ is defined as follows. The angle ϕ parametrizes the $SO(2) \subset O(2)$ transformation

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} , \qquad 0 \le \phi < 2\pi \; . \tag{2.64}$$

Under the $SO(2)$ transformation (2.8), the bosonic coordinates transform as

$$(w_1, w_2) \rightarrow (w_1^\phi, w_2^\phi) \equiv (w_1, w_2) \cdot R_\phi \tag{2.65a}$$

$$(a_3, a'_{12}) \rightarrow (a_3^\phi, a_{12}'^\phi) \equiv (a_3, a'_{12}) \cdot R_{2\phi} \tag{2.65b}$$

$$a_0 \rightarrow a_0^\phi \equiv a_0 \tag{2.65c}$$

and likewise for their corresponding fermionic superpartners in $\mathcal{M}$. Consequently[7]

$$\Sigma(a_3, a_0, w_1, w_2) \rightarrow \Sigma^\phi \equiv \Sigma(a_3^\phi, a_0, w_1^\phi, w_2^\phi) \; . \tag{2.66}$$

Since Σ transforms nontrivially under $O(2)$, an "$O(2)$-gauge-fixing" prescription is locally equivalent to a particular functional choice for Σ, but the *global* equivalence is not guaranteed [10]: If, on the one hand, the $O(2)$ redundancy group is broken completely, then each point on the ADHM manifold is indeed in 1-to-1 correspondence with a physical 2-instanton configuration. On the other hand, it generically happens that specifying Σ does not break $O(2)$ completely, but leaves a residual *discrete* subgroup $G_2 \subset O(2)$. The

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[7] This is an interesting self-referential equation since a_3^ϕ depends on a'_{12} which, in turn, depends on Σ.

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action of G_2 must then be modded out in constructing the physical measure $d\mu^{(2)}_{\mathrm{phys}}$; this is precisely the overall factor [10]

$$\mathcal{S}_2 \;=\; \dim G_2 \qquad\qquad (2.67)$$

that appears in Eq. (2.55). Intuitively, G_2 will include the permutation group $P(2)$ which exchanges the identities of the two instantons. In practice, G_2 is often much bigger than $P(2)$. For instance, for the simple choice $\Sigma \equiv 0$, G_2 is the dihedral group D_8, and consequently $\mathcal{S}_2 = \dim(D_8) = 16$ [11]; other discrete groups can occur for different choices of Σ.

The remaining quantity introduced above that needs to be specified is the parameter C_2 in Eq. (2.63); in our conventions it contains all collective-coordinate-independent multiplicative factors in (2.55) apart from $\mathcal{S}_2^{-1}$. In fact C_2 is the same parameter as entered the $O(2)$-invariant measure, (2.23) and (2.54). To see this, it suffices to consider the limit $|a'_{22}| \to \infty$. For nonpathological Σ this means

$$\frac{\left| |a_3|^2 \;-\; |a'_{12}|^2 \;-\; \tfrac{1}{8}\tfrac{d\Sigma^\phi}{d\phi}\big|_{\phi=0} \right|}{|a_3|^2} \;\longrightarrow\; 1 \qquad\qquad (2.68)$$

so that the clustering property forces

$$C_2 \;=\; (C_1)^2 \qquad\qquad (2.69)$$

as per Eq. (2.54) above.

Now let us, by hand, restore $O(2)$ invariance to the $O(2)$-gauge-fixed measure (2.55). The first step is to "integrate in" the off-diagonal elements a'_{12} and $\mathcal{M}'_{12}$ by inserting the factors of unity

$$1 \;=\; 16|a_3|^4 \int d^4a'_{12} \prod_{c=1,2,3} \delta\big(\tfrac{1}{4}\mathrm{tr}_2\,\tau^c[(\bar{a}a)_{1,2}-(\bar{a}a)_{2,1}]\big)\delta\big(\bar{a}_3 a'_{12}+\bar{a}'_{12}a_3-\tfrac{1}{2}\Sigma(a_3,a_0,w_1,w_2)\big)$$

$$(2.70)$$

and

$$1 \;=\; \frac{1}{4|a_3|^2}\int d^2\mathcal{M}'_{12}\,\delta^2\big((\bar{a}\mathcal{M})_{1,2}-(\bar{a}\mathcal{M})_{2,1}\big) \qquad\qquad (2.71)$$

The second δ-function in (2.70) enforces the "$O(n)$-gauge-fixing" (2.61); the other two δ-functions in (2.70)-(2.71) are the by-now-familiar implementations of the constraints (2.5a) and (2.13a). Next one performs a change of dummy integration variables $a \to a^\phi$ and $\mathcal{M} \to \mathcal{M}^\phi$ as defined by Eqs. (2.65). The δ-function in (2.71), and the first δ-function in (2.70), are unchanged, since their arguments are formed from the $O(2)$-invariant matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Likewise H is an invariant. The only non-invariants are the second δ-function in (2.70), and the piece $\left| |a_3|^2 - |a'_{12}|^2 - \frac{1}{8}(d\Sigma^\phi/d\phi)|_{\phi=0} \right|$ from $J_{\text{bose}}^{1/2}$. Together they transform nontrivially, as follows:

$$\left| |a_3|^2 - |a'_{12}|^2 - \frac{1}{8}\frac{d\Sigma^\phi}{d\phi}\Big|_{\phi=0} \right| \cdot \delta\big(\bar{a}_3 a'_{12} + \bar{a}'_{12} a_3 - \frac{1}{2}\Sigma(a_3, a_0, w_1, w_2)\big) \tag{2.72}$$
$$\longrightarrow \frac{1}{4}\big|Z'(\phi)\big|\,\delta\big(Z(\phi)\big)$$

where

$$Z(\phi) = \bar{a}_3^\phi a'^\phi_{12} + \bar{a}'^\phi_{12} a_3^\phi - \tfrac{1}{2}\Sigma^\phi \tag{2.73}$$

in the notation of Eq. (2.65). Inserting another factor of unity, namely

$$1 = \frac{1}{2\pi}\int_0^{2\pi} d\phi \tag{2.74}$$

and integrating over the right-hand side of (2.72) gives a factor of

$$\frac{1}{2\pi}\cdot\frac{1}{4}\cdot k \tag{2.75}$$

where the integer k counts the number of times in the interval $[0, 2\pi]$ that $Z(\phi)$ vanishes. Obviously k is related to the existence of the discrete subgroup G_2 discussed above; in fact one typically has

$$k = \tfrac{1}{2}\mathcal{S}_2 \tag{2.76}$$

where the factor of $\frac{1}{2}$ is due to the fact that in Eqs. (2.74) and (2.64) we are only integrating over half the $O(2)$ group, namely $SO(2)$.[8]

Combining Eqs. (2.55), (2.63), (2.69)-(2.71), and (2.75)-(2.76) gives, finally,

$$\int d\mu^{(2)}_{\text{phys}} = \frac{(C_1)^2}{4\pi}\int \prod_{i=1}^2 d^4 w_i\, d^2\mu_i \prod_{(ij)_2} d^4 a'_{ij}\, d^2\mathcal{M}'_{ij}$$
$$\times \prod_{c=1,2,3} \delta\big(\tfrac{1}{4}\text{tr}_2\,\tau^c[(\bar{a}a)_{1,2} - (\bar{a}a)_{2,1}]\big)\,\delta^2\big((\bar{a}\mathcal{M})_{1,2} - (\bar{a}\mathcal{M})_{2,1}\big)\ . \tag{2.77}$$

Notice that the Σ-dependent discrete group factor $\mathcal{S}_2$ introduced in Eq. (2.55) has canceled out. In fact, since $4\pi = \text{Vol}(O(2))$, we have recaptured precise agreement with the earlier $O(2)$- and SUSY-invariant expression (2.23) and (2.54), as promised. This concludes our list of checks of the proposed $N = 1$ measure (2.23) and (2.54).

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[8] The relation (2.76) may fail if Σ is taken to depend on odd powers of the w_i (breaking the "parity" symmetry $w_i \to -w_i$). If so, then rather than Eq. (2.74) one should integrate over both halves of the $O(2)$ group separately and divide the sum by 4π. The final answer for $d\mu^{(2)}_{\text{phys}}$ is then the same.

3. The $N = 2$ supersymmetric collective coordinate integration measure

3.1. ADHM and $N = 2$ SUSY review

Next we turn to the $N = 2$ case. (The extension to $N = 4$ is more intricate still and will be discussed in a separate publication.) The particle content of $N = 2$ supersymmetric Yang-Mills theory comprises, in addition to the gauge field v_m and gaugino λ_α, an adjoint complex Higgs A and Higgsino ψ_α. The fermion zero modes of ψ_α are defined in analogy with Eq. (2.11):

$$(\psi_\alpha)^{\dot\beta}_{\ \dot\gamma} \;=\; \bar U^{\dot\beta\gamma} \mathcal{N}_\gamma f \,\bar b\, U_{\alpha\dot\gamma} \;-\; \bar U^{\dot\beta}_{\ \alpha}\, bf \mathcal{N}^{\gamma T} U_{\gamma\dot\gamma} \;, \tag{3.1}$$

where the Weyl-spinor-valued matrix

$$\mathcal{N}^\gamma \;=\; \begin{pmatrix} \nu_1^\gamma & \cdots & \nu_n^\gamma \\ & \mathcal{N}'^\gamma & \end{pmatrix} \tag{3.2}$$

satisfies linear constraints analogous to Eq. (2.13):

$$\bar a^{\dot\alpha\gamma}\mathcal{N}_\gamma \;=\; -\mathcal{N}^{\gamma T} a_\gamma^{\ \dot\alpha} \;, \tag{3.3a}$$

$$\mathcal{N}'^T_\gamma \;=\; \mathcal{N}'_\gamma \;. \tag{3.3b}$$

Under $O(n)$, $\mathcal{N}$ transforms like $\mathcal{M}$ and $\Delta(x)$; see Eq. (2.8).

The construction of the classical Higgs A is more involved, and goes as follows [11]. A has the additive form $A = A^{(1)} + A^{(2)}$, where

$$i\, A^{(1)\dot\alpha}_{\ \ \dot\beta} \;=\; \frac{1}{2\sqrt 2}\, \bar U^{\dot\alpha\alpha}\big(\mathcal{N}_\alpha f \mathcal{M}^{\beta T} - \mathcal{M}_\alpha f \mathcal{N}^{\beta T}\big) U_{\beta\dot\beta} \;, \tag{3.4}$$

and $i\, A^{(2)\dot\alpha}_{\ \ \dot\beta} = \bar U^{\dot\alpha\alpha}\, \mathcal{A}_\alpha^{\ \beta} U_{\beta\dot\beta}$, with $\mathcal{A}$ a block-diagonal constant matrix,

$$\mathcal{A}_\alpha^{\ \beta} \;=\; \begin{pmatrix} \mathcal{A}_{00\alpha}^{\ \ \beta} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{A}_{\text{tot}}\,\delta_\alpha^{\ \beta} & \\ 0 & & & \end{pmatrix} \;. \tag{3.5}$$

$\mathcal{A}_{00}$ is related in a trivial way to the adjoint complex VEV v (which we point in the τ^3 direction),

$$\mathcal{A}_{00\alpha}^{\ \ \beta} \;=\; \tfrac{i}{2}\, \mathrm{v}\, \tau^3{}_\alpha^{\ \beta} \;, \qquad \bar{\mathcal{A}}_{00\alpha}^{\ \ \beta} \;=\; -\tfrac{i}{2}\, \bar{\mathrm{v}}\, \tau^3{}_\alpha^{\ \beta} \;, \tag{3.6}$$

The $n \times n$ antisymmetric matrix $\mathcal{A}_{\text{tot}}$ is defined as the solutions to the inhomogeneous linear equation

$$\mathbf{L} \cdot \mathcal{A}_{\text{tot}} \;=\; \Lambda + \Lambda_f \;, \tag{3.7}$$

where Λ and Λ_f are the $n \times n$ antisymmetric matrices

$$\Lambda_{lk} = \bar{w}_l \mathcal{A}_{00} w_k - \bar{w}_k \mathcal{A}_{00} w_l \tag{3.8}$$

and

$$\Lambda_f = \frac{1}{2\sqrt{2}} \left(\mathcal{M}^{\beta T} \mathcal{N}_\beta - \mathcal{N}^{\beta T} \mathcal{M}_\beta \right) . \tag{3.9}$$

$\mathbf{L}$ is a linear operator that maps the space of $n \times n$ scalar-valued antisymmetric matrices onto itself. Explicitly, if Ω is such a matrix, then $\mathbf{L}$ is defined as [11]

$$\mathbf{L} \cdot \Omega = \tfrac{1}{2}\{\Omega, W\} - \tfrac{1}{2}\mathrm{tr}_2 \left([\bar{a}', \Omega]a' - \bar{a}'[a', \Omega] \right) \tag{3.10}$$

where W is the symmetric scalar-valued $n \times n$ matrix

$$W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k \tag{3.11}$$

From Eqs. (3.7)-(3.11) one sees that $\mathcal{A}_{\mathrm{tot}}$ transforms in the adjoint representation of $O(n)$ (i.e., like a', $\mathcal{M}'$ and $\mathcal{N}'$).

As shown in [11], defined in this way, the Higgs field A correctly satisfies the classical Euler-Lagrange equation

$$\mathcal{D}^2 A = \sqrt{2}\, i\, [\lambda, \psi] \tag{3.12}$$

where $\mathcal{D}^2$ is the covariant Klein-Gordon operator in the multi-instanton background, and λ and ψ are given by (2.11) and (3.1), respectively.

As in the $N = 1$ case, the $N = 2$ supersymmetry algebra may be realized directly on these collective coordinates. Under the action of $\sum_{i=1,2} \xi_i Q_i + \bar{\xi}_i \bar{Q}_i$ one has [5]:[9]

$$\delta a_{\alpha\dot\alpha} = \bar{\xi}_{1\dot\alpha} \mathcal{M}_\alpha + \bar{\xi}_{2\dot\alpha} \mathcal{N}_\alpha \tag{3.13a}$$

$$\delta \mathcal{M}_\gamma = -4ib\xi_{1\gamma} - 2\sqrt{2}\, \mathcal{C}_{\gamma\dot\alpha} \bar{\xi}_2^{\dot\alpha} \tag{3.13b}$$

$$\delta \mathcal{N}_\gamma = -4ib\xi_{2\gamma} + 2\sqrt{2}\, \mathcal{C}_{\gamma\dot\alpha} \bar{\xi}_1^{\dot\alpha} \tag{3.13c}$$

$$\delta \mathcal{A}_{\mathrm{tot}} = 0 \tag{3.13d}$$

---

[9] As noted in [5], actually the $N = 2$ algebra is not faithfully represented by Eqs. (3.13). For instance, the anticommutator $\{\bar{Q}_1, \bar{Q}_2\}$, rather than vanishing when acting on a, $\mathcal{M}$ or $\mathcal{N}$, gives a residual $O(n)$ symmetry transformation of the form (2.8). (This is analogous to naive realizations of supersymmetry that fail to commute with Wess-Zumino gauge fixing, for example.) For present purposes this poses no problem, as we are always ultimately concerned with $O(n)$ singlets; otherwise one would have to covariantize the supersymmetry transformations with respect to $O(n)$ in the standard way. This is also the reason why Eq. (3.15) below is not symmetric in $\bar{\theta}_1 \rightleftharpoons \bar{\theta}_2$.

Here $\mathcal{C}_{\gamma\dot\alpha}$ is the $(n+1) \times n$ quaternion-valued matrix

$$\mathcal{C} = \begin{pmatrix} \mathcal{A}_{00}w_1 - w_k\mathcal{A}_{\text{tot}\,k1} & \cdots & \mathcal{A}_{00}w_n - w_k\mathcal{A}_{\text{tot}\,kn} \\ & [\mathcal{A}_{\text{tot}}\,,\,a'] & \end{pmatrix} . \tag{3.14}$$

Of course $\mathcal{A}_{\text{tot}}$ is not an independent collective coordinate since the invertible linear equation (3.7) fixes it in terms of a, $\mathcal{M}$, $\mathcal{N}$, and the VEV v; nevertheless we will find that the formalism simplifies when $\mathcal{A}_{\text{tot}}$ is treated as though it were independent.

As in the $N = 1$ case, it is illuminating to promote a to a space-time-constant $N = 2$ superfield $a(\bar\theta_1, \bar\theta_2)$ [5]:

$$\begin{aligned} a_{\alpha\dot\alpha} \longrightarrow a_{\alpha\dot\alpha}(\bar\theta_i) &= e^{\bar\theta_2 \bar{Q}_2} \times e^{\bar\theta_1 \bar{Q}_1} \times a_{\alpha\dot\alpha} \\ &= a_{\alpha\dot\alpha} + \bar\theta_{1\dot\alpha}\mathcal{M}_\alpha + \bar\theta_{2\dot\alpha}\mathcal{N}_\alpha + 2\sqrt{2}\,\mathcal{C}_{\alpha\dot\beta}\bar\theta_2^{\dot\beta}\bar\theta_{1\dot\alpha} + \sqrt{2}\,\bar\theta_{1\dot\alpha}\bar\theta_2^2 \mathcal{C}_{\mathcal{N}\alpha} \end{aligned} \tag{3.15}$$

where the Grassmann matrix $\mathcal{C}_{\mathcal{N}}$ is defined in analogy with $\mathcal{C}$,

$$\mathcal{C}_{\mathcal{N}} = \begin{pmatrix} \mathcal{A}_{00}\nu_1 - \nu_k\mathcal{A}_{\text{tot}\,k1} & \cdots & \mathcal{A}_{00}\nu_n - \nu_k\mathcal{A}_{\text{tot}\,kn} \\ & [\mathcal{A}_{\text{tot}}\,,\,\mathcal{N}'] & \end{pmatrix} . \tag{3.16}$$

The $N = 2$ supersymmetric ADHM constraint then reads [5]:

$$\bar{a}(\bar\theta_i)a(\bar\theta_i) = \big(\bar{a}(\bar\theta_i)a(\bar\theta_i)\big)^T \propto \delta^{\dot\beta}{}_{\dot\alpha} . \tag{3.17}$$

Indeed the constant component of (3.17) is the bosonic constraint (2.5a) while the $\bar\theta_1$ and $\bar\theta_2$ components are the fermionic constraints (2.13a) and (3.3a), respectively. The $\bar\theta_{1\dot\alpha}\bar\theta_{2\dot\beta}$ component of (3.17) gives

$$\Lambda_f = \bar{\mathcal{C}}a - (\bar{\mathcal{C}}a)^T , \tag{3.18}$$

which appears to be new, but is actually just a concise rewriting of Eq. (3.7). The remaining $\bar\theta_i$ components of (3.17) turn out to be "auxiliary" as they contain no new information. Some are satisfied trivially, while others boil down to linear combinations of the previous relations.

3.2. Ansatz for the $N = 2$ measure

For arbitrary topological number n we can now write down what we assert to be the unique $N = 2$ supersymmetric collective coordinate measure which respects criteria (i)-(iii)

listed in Sec. 2.2 above:

$$
\begin{aligned}
\int d\mu_{\text{phys}}^{(n)} &\equiv \frac{1}{\text{Vol}(O(n))} \int d\mu^{(n)} \\
&= \frac{C_n'}{\text{Vol}(O(n))} \int \prod_{i=1}^{n} d^4 w_i d^2 \mu_i d^2 \nu_i \prod_{(ij)_n} d^4 a_{ij}' d^2 \mathcal{M}_{ij}' d^2 \mathcal{N}_{ij}' \prod_{\langle ij \rangle_n} d(\mathcal{A}_{\text{tot}})_{i,j} \\
&\quad \times \prod_{\langle ij \rangle_n} \prod_{c=1,2,3} \delta\big(\tfrac{1}{4}\text{tr}_2\, \tau^c [(\bar{a}a)_{i,j} - (\bar{a}a)_{j,i}]\big)\, \delta^2\big((\bar{a}\mathcal{M})_{i,j} - (\bar{a}\mathcal{M})_{j,i}\big) \\
&\qquad \times\ \delta^2\big((\bar{a}\mathcal{N})_{i,j} - (\bar{a}\mathcal{N})_{j,i}\big)\, \delta\big((\mathbf{L}\cdot\mathcal{A}_{\text{tot}} - \Lambda - \Lambda_f)_{i,j}\big)\ .
\end{aligned}
\tag{3.19}
$$

This expression appears to depend on the VEV v which enters the definition of Λ, but this dependence is fictitious, since the $\mathcal{A}_{\text{tot}}$ integration just gives $\det^{-1}\mathbf{L}$ independent of v. Thus we could instead rewrite the final δ-function simply as $\delta((\mathbf{L}\cdot\mathcal{A}_{\text{tot}})_{i,j})$ but we prefer to maintain a parallel treatment of the spin-0, spin-$\frac{1}{2}$ and spin-1 constraints (3.7), (2.13a) and (2.5a). It is for this reason that we treat $\mathcal{A}_{\text{tot}}$ as an independent collective coordinate in (3.19) rather than eliminate it using (3.7). We will solve for the constants C_n' in Sec. 3.4 below, using the clustering criterion (iv) as before.

In the 1-instanton sector all these constraints disappear, and Eq. (3.19) collapses to the known 't Hooft measure [1]

$$
\int d\mu_{\text{phys}}^{(1)} \equiv \tfrac{1}{2}\int d\mu^{(1)} = \tfrac{1}{2}C_1' \int d^4 w\, d^2\mu\, d^2\nu\, d^4 a'\, d^2\mathcal{M}'\, d^2\mathcal{N}'\ ,
\tag{3.20}
$$

where $\frac{1}{2}C_1'$ is the appropriate scheme-dependent multiplicative factor.

3.3. $O(n)$ invariance, $N = 2$ SUSY invariance, and dimensional power counting

As in the $N = 1$ case, the $O(n)$ invariance of this measure is obvious by inspection. As for $N = 2$ supersymmetry invariance under (3.13), this too is obvious for $\sum_{i=1,2}\xi_i Q_i$. For $\sum_{i=1,2}\bar{\xi}_i\bar{Q}_i$ one can show that the arguments of the four δ-functions in (3.19) transform into linear combinations of one another, and furthermore that the associated transformation matrix has superdeterminant unity, so that $N = 2$ supersymmetry is guaranteed. As in the $N = 1$ case this feature stems from the observation that these four constraints assemble into a single $N = 2$ multiplet as per Eq. (3.17).

Next we verify the counting requirement (iii) given in Sec. 2.2. Of the $2n^2 + 6n$ fermionic modes in (3.19), $2n(n-1)$ of them are saturated by the second and third δ-functions, leaving $8n$ unbroken adjoint fermion zero modes as required. Note that the final δ-function does not lift any fermionic zero modes despite the presence of the fermion bilinear Λ_f; this is because the $\mathcal{A}_{\text{tot}}$ integration yields the purely bosonic quantity $\det^{-1}\mathbf{L}$

as noted above. As in the $N = 1$ theory, the bosonic dimensionality of (3.19) scales, correctly, like

$$[a]^{(2n^2+6n)-(3n^2-3n)+(n^2-n)+(n^2-n)-(n^2-n)} \sim [a]^{8n} , \qquad (3.21)$$

the five terms in the exponent coming, respectively, from the w_i and a'_{ij} integration variables, and from the four δ-functions. (We do not count factors of $\mathcal{A}_{\text{tot}}$ here as they cancel between the differentials and the δ-functions.)

3.4. Cluster decomposition

Next we check cluster decomposition in the limit $|a'_{nn}| \to \infty$. The calculation proceeds just as in the $N = 1$ case, except that prior to eliminating the $\mathcal{M}'_{in}$ one eliminates the $(\mathcal{A}_{\text{tot}})_{in}$ via

$$\int \prod_{i=1}^{n-1} d(\mathcal{A}_{\text{tot}})_{in} \, \delta\big((\mathbf{L} \cdot \mathcal{A}_{\text{tot}} - \Lambda - \Lambda_f)_{i,n} \big) \times \cdots \qquad (3.22)$$

The argument of the δ-function may be expanded as

$$\big(\mathbf{L} \cdot \mathcal{A}_{\text{tot}} - \Lambda - \Lambda_f \big)_{i,n} = |a'_{nn}|^2 (\mathcal{A}_{\text{tot}})_{in} + \cdots \qquad (3.23)$$

neglecting subleading terms. Thus these quantities scale like

$$(\mathcal{A}_{\text{tot}})_{in} \sim |a'_{nn}|^{-2} \qquad (3.24)$$

and Eq. (3.22) yields a Jacobian factor

$$J_{\mathcal{A}_{\text{tot}}} = \left(\frac{1}{|a'_{nn}|^2} \right)^{n-1} + \cdots, \qquad (3.25)$$

again neglecting subleading terms. The other new feature for the $N = 2$ case is the existence of the Higgsino collective coordinates $\mathcal{N}$. The elimination of the $\mathcal{N}'_{in}$ proceeds identically to Eqs. (2.38)-(2.40) for the $\mathcal{M}'_{in}$, and likewise gives

$$J_{\mathcal{N}} = |a'_{nn}|^{2(n-1)} + \cdots. \qquad (3.26)$$

The remaining integrations are just those of the $N = 1$ model. Note that the factors of $|a'_{nn}|$ again cancel among Eqs. (2.40), (2.44), (2.46), (3.25) and (3.26).

It still must be checked that as $|a'_{nn}| \to \infty$ the arguments of the remaining δ-functions in (3.19) properly reduce to those of the $(n-1)$-instanton case, i.e., are built from the truncated matrices $\tilde{a}$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{A}}_{\text{tot}}$ (see Eq. (2.49)). With the scaling relations (2.52)

and (3.24) this is easily verified, up to corrections of order $1/|a'_{nn}|^2$. As in the $N = 1$ case, the clustering condition (2.36) then gives, simply,

$$C'_n = (C'_1)^n .$$
(3.27)

3.5. Agreement in the 2-instanton sector

Still paralleling Sec. 2, we now check that in the 2-instanton sector, our $O(n)$- and $N = 2$ SUSY-invariant collective coordinate measure (3.19) and (3.27) is equivalent to the known first-principles $O(n)$-gauge-fixed measure [11,10]

$$\int d\mu^{(2)}_{\text{phys}} = \frac{1}{\mathcal{S}_2} \int d^4 w_1 d^4 w_2 d^4 a'_{11} d^4 a'_{22} d^2 \mu_1 d^2 \mu_2 d^2 \mathcal{M}'_{11} d^2 \mathcal{M}'_{22} d^2 \nu_1 d^2 \nu_2 d^2 \mathcal{N}'_{11} d^2 \mathcal{N}'_{22}$$
$$\times \left(J_{\text{bose}}/J_{\text{fermi}} \right)^{1/2} .$$
(3.28)

The chief difference with the $N = 1$ case is that now J_{fermi} is the square of Eq. (2.56) [11], there being twice as many adjoint fermions in the $N = 2$ model. Consequently Eq. (2.63) is modified to

$$\left(J_{\text{bose}}/J_{\text{fermi}} \right)^{1/2} = C'_2 \frac{\left| |a_3|^2 - |a'_{12}|^2 - \frac{1}{8} \frac{d\Sigma^\phi}{d\phi} \big|_{\phi=0} \right|}{H} ,$$
(3.29)

where cluster still fixes

$$C'_2 = (C'_1)^2$$
(3.30)

as before. Now the insertions of unity into (3.28) comprise, not just Eqs. (2.70), (2.71), and (2.74), but additionally

$$1 = \frac{1}{4|a_3|^2} \int d^2 \mathcal{N}'_{12} \, \delta^2 \big((\bar{a}\mathcal{N})_{1,2} - (\bar{a}\mathcal{N})_{2,1} \big)$$
(3.31)

and

$$1 = \int d(\mathcal{A}_{\text{tot}})_{12} \, \delta \big((\mathbf{L} \cdot \mathcal{A}_{\text{tot}} - \Lambda - \Lambda_f)_{1,2} \big) \det \mathbf{L} .$$
(3.32)

Recall that $\mathbf{L}$ is a $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ dimensional linear operator on the space of $n \times n$ antisymmetric matrices. For $n = 2$ this space is 1-dimensional, and $\mathbf{L}$ is simply the scalar H:

$$\det \mathbf{L} = \mathbf{L} = H .$$
(3.33)

So the factor of $\det \mathbf{L}$ in Eq. (3.32) cancels the denominator in Eq. (3.29). The rest of the argument goes through precisely as in Sec. 2.5, and once again gives exact equivalence to the postulated $O(2)$- and SUSY-invariant form (3.19) and (3.27).

4. Incorporation of matter

The $N = 1$ and $N = 2$ multi-instanton measures detailed above for pure $SU(2)$ gauge theory are easily extended to incorporate fundamental matter multiplets. $N = 1$ supersymmetric fundamental matter in the n-instanton sector is discussed in Appendix C of [5], and in [20]. For application to the Section to follow, we will focus here on the $N = 2$ case, and consider adding N_F matter hypermultiplets which transform in the fundamental representation of $SU(2)$. Each $N = 2$ hypermultiplet corresponds to a pair of $N = 1$ chiral multiplets, Q_i and $\tilde{Q}_i$ where $i = 1, 2, \cdots, N_F$, which contain scalar quarks q_i and $\tilde{q}_i$ respectively and fermionic partners χ_i and $\tilde{\chi}_i$. The fundamental fermion zero modes associated with χ_i and $\tilde{\chi}_i$ were constructed in [17,21]:

$$(\chi_i^\alpha)^{\dot\beta} = \bar{U}_\lambda^{\dot\beta\alpha} b_{\lambda k} f_{kl} \mathcal{K}_{li} , \quad (\tilde{\chi}_i^\alpha)^{\dot\beta} = \bar{U}_\lambda^{\dot\beta\alpha} b_{\lambda k} f_{kl} \tilde{\mathcal{K}}_{li} \qquad (4.1)$$

with α a Weyl index and $\dot\beta$ an $SU(2)$ color index. Here each $\mathcal{K}_{ki}$ and $\tilde{\mathcal{K}}_{ki}$ is a Grassmann number rather than a Grassmann spinor; there is no $SU(2)$ index. The normalization matrix of these modes is given by [21]

$$\int d^4x (\chi_i^\alpha)_{\dot\beta} (\tilde{\chi}_{\alpha j})^{\dot\beta} = \pi^2 \mathcal{K}_{li} \tilde{\mathcal{K}}_{lj} \qquad (4.2)$$

so that the hypermultiplet part of the n-instanton measure reads [5]

$$\int d\mu_{\rm hyp}^{(n)} = \frac{1}{\pi^{2nN_F}} \int \prod_{i=1}^{N_F} d\mathcal{K}_{1i} \cdots d\mathcal{K}_{ni} \, d\tilde{\mathcal{K}}_{1i} \cdots d\tilde{\mathcal{K}}_{ni} . \qquad (4.3)$$

The total measure is then simply

$$d\mu_{\rm phys}^{(n)} \times d\mu_{\rm hyp}^{(n)} . \qquad (4.4)$$

One can also consider the case of coupling to a single (massive) $N = 2$ matter hypermultiplet in the adjoint representation of the gauge group, but this is best understood by starting from the $N = 4$ theory and will be discussed in a separate publication.

5. Explicit expressions for the Seiberg-Witten prepotentials, and for the 4-derivative/8-fermion term

In earlier work [5] we presented a general formula for the $\mathcal{F}_n$ (i.e., the n-instanton contribution to the Seiberg-Witten prepotentials [3,4]) as integrations of the exponentiated multi-instanton action over the space of $N = 2$ supersymmetric collective coordinates. However, at the time, the collective coordinate integration measure was not known. For

the purposes of self-containedness, we repeat those expressions here, inserting the explicit expression for the $N = 2$ measure, Eqs. (3.19) and (3.27).

We start with the case of pure $N = 2$ supersymmetric $SU(2)$ gauge theory [3,11]. In this model the n-instanton action reads[10] [11]

$$S_{\text{inst}}^0 = \frac{8n\pi^2}{g^2} + 16\pi^2|\mathcal{A}_{00}|^2 \sum_{k=1}^{n} |w_k|^2 - 8\pi^2 \operatorname{Tr}_n \mathcal{A}_{\text{tot}}\bar{\Lambda} + 4\sqrt{2}\,\pi^2 \mu_k^\alpha \,\bar{\mathcal{A}}_{00\alpha}{}^\beta \,\nu_{k\beta} \,, \quad (5.1)$$

where the notation is that of Eqs. (2.6), (2.12), (3.2), and (3.6)-(3.8) above. As shown in [5], this may be rewritten as a manifestly $N = 2$ supersymmetric "F-term" :

$$S_{\text{inst}}^0 = \frac{8n\pi^2}{g^2} - \pi^2 \operatorname{Tr} \bar{a}(\bar{\theta}_i)\big(\mathcal{P}_\infty + 1\big)a(\bar{\theta}_i)\Big|_{\bar{\theta}_1^2\bar{\theta}_2^2} \,. \quad (5.2)$$

Here the capitalized 'Tr' indicates a trace over both ADHM and $SU(2)$ indices, $\mathcal{P}_\infty$ denotes the $(n+1) \times (n+1)$ matrix

$$\mathcal{P}_\infty \;=\;=\; 1 - b\bar{b} \;=\; \delta_{\lambda 0}\delta_{\kappa 0} \,, \quad (5.3)$$

and $a(\bar{\theta}_i)$ is the space-time-constant "superfield" given in Eq. (3.15).

As for the measure, it is useful to factor the physical $N = 2$ measure, Eq. (3.19) and (3.27), as follows:

$$\int d\mu_{\text{phys}}^{(n)} \;=\; \int d^4x_0 \, d^2\xi_1 \, d^2\xi_2 \int d\tilde{\mu}_{\text{phys}}^{(n)} \quad (5.4)$$

where (x_0, ξ_1, ξ_2) gives the global position of the multi-instanton in $N = 2$ superspace. Explicitly, x_0, ξ_1 and ξ_2 are the linear combinations proportional to the 'trace' components of the $n \times n$ matrices a', $\mathcal{M}'$ and $\mathcal{N}'$, respectively [11]:

$$x_0 \;=\; \frac{1}{n}\operatorname{Tr}_n a' \,, \quad \xi_1 \;=\; \frac{1}{4n}\operatorname{Tr}_n\mathcal{M}' \,, \quad \xi_2 \;=\; \frac{1}{4n}\operatorname{Tr}_n\mathcal{N}' \,. \quad (5.5)$$

Note that these $N = 2$ superspace modes do not enter into the δ-function constraints in (3.19) and so do indeed factor out in this simple way. Furthermore, the four exact supersymmetric modes $\xi_{1\alpha}$ and $\xi_{2\alpha}$ are the only fermionic modes that are not lifted by (i.e., do not appear in) the action (5.1).

In terms of these quantities, the exact all-instanton-orders expression for the Seiberg-Witten prepotential reads:

$$\mathcal{F}(\text{v}) \;\equiv\; \mathcal{F}_{\text{pert}}(\text{v}) + \mathcal{F}_{\text{inst}}(\text{v}) \;=\; \frac{i}{2\pi}\text{v}^2 \log\frac{2\text{v}^2}{e^3\Lambda^2} - \frac{i}{\pi}\sum_{n=1}^{\infty} \mathcal{F}_n \left(\frac{\Lambda}{\text{v}}\right)^{4n} \text{v}^2 \,. \quad (5.6)$$

---

[10] This expression contrasts with the $N = 1$ supersymmetric action, which in ADHM variables looks like the noninteracting sum of n single instantons; see Appendix C of [5], and Ref. [20].

Here Λ is the dynamically generated scale in the Pauli-Villars scheme, and the $\mathcal{F}_n$ may be expressed as the following explicit finite-dimensional collective coordinate integrals [5]:

$$\mathcal{F}(\mathrm{v})\Big|_{n\text{-inst}} \equiv -\frac{i\mathcal{F}_n}{\pi}\left(\frac{\Lambda}{\mathrm{v}}\right)^{4n}\mathrm{v}^2 = 8\pi i \int d\tilde{\mu}_{\mathrm{phys}}^{(n)} \exp(-S_{\mathrm{inst}}^0) . \tag{5.7}$$

Next we consider the class of models in which $N=2$ supersymmetric $SU(2)$ gauge theory is coupled to N_F flavors of massless quark hypermultiplets [4,5]. In these cases the n-instanton action reads [5]

$$S_{\mathrm{inst}}^{N_F} = S_{\mathrm{inst}}^0 - 8\pi^2 \mathrm{Tr}_n \Lambda_{\mathrm{hyp}} \mathcal{A}_{\mathrm{tot}} , \tag{5.8}$$

where Λ_{hyp} is the $n \times n$ antisymmetric matrix

$$(\Lambda_{\mathrm{hyp}})_{k,l} = \frac{i\sqrt{2}}{16}\sum_{i=1}^{N_F}\left(\mathcal{K}_{ki}\tilde{\mathcal{K}}_{li} + \tilde{\mathcal{K}}_{ki}\mathcal{K}_{li}\right) , \tag{5.9}$$

and $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are the fundamental fermion collective coordinates defined in Eq. (4.1). In these models the all-instanton-orders contributions to the prepotential are still correctly given by Eq. (5.7), with the substitutions

$$S_{\mathrm{inst}}^0 \rightarrow S_{\mathrm{inst}}^{N_F} , \qquad d\tilde{\mu}_{\mathrm{phys}}^{(n)} \rightarrow d\tilde{\mu}_{\mathrm{phys}}^{(n)} \times d\mu_{\mathrm{hyp}}^{(n)} \tag{5.10}$$

with $d\mu_{\mathrm{hyp}}^{(n)}$ as in Eq. (4.3). Actually for $N_F > 0$ all odd-instanton contributions vanish in the massless case due to a discrete $\mathbb{Z}_2$ symmetry [4,5]. The incorporation of hypermultiplet masses is straightforward but rather too lengthy to recapitulate here; see Ref. [5] for a discussion of the $\mathcal{F}_n$ in this case, and (say) Sec. 2.1 of [14] for the full expression for the perturbative part of $\mathcal{F}$ (which is often incompletely rendered in the literature).

Given these expressions for the prepotential, one also knows the all-instanton-orders expansion of the quantum modulus $u = \langle \mathrm{Tr}\Phi^2\rangle$, since on general grounds [22-23]

$$u(\mathrm{v})\Big|_{n\text{-inst}} = 2i\pi n \cdot \mathcal{F}(\mathrm{v})\Big|_{n\text{-inst}} . \tag{5.11}$$

The above collective coordinate integral expressions for $\mathcal{F}$ and u constitute a closed solution, in quadratures, of the low-energy dynamics of the Seiberg-Witten models. It is interesting that this solution is obtained purely from the semiclassical regime, without appeal to electric-magnetic duality. For $N_F \leq 3$ this may be regarded as academic, in light of the exact results of [3,4]. However, for the conformally invariant case $N_F = 4$, the

multi-instanton solution contains information not present in [4], namely the all-instanton-orders relation between the microscopic ($SU(2)$) and effective ($U(1)$) complexified coupling constants, τ_{micro} and τ_{eff}. For $N_F = 4$, Eq. (5.6) should simply be replaced by

$$\mathcal{F}(\text{v}) \;=\; \tfrac{1}{4}\,\tau_{\text{micro}}\,\text{v}^2 \;-\; \frac{i}{\pi}\,\sum_{n=0,2,4,\cdots}\mathcal{F}_n\,q^n\,\text{v}^2 \tag{5.12}$$

where $q = \exp(i\pi\tau_{\text{micro}})$, and the $\mathcal{F}_n$ have the same collective coordinate integral representation as for $N_F < 4$. By definition, the effective coupling τ_{eff} is obtained from the prepotential via $\tau_{\text{eff}} = 2\mathcal{F}''(\text{v})$. The constants $\mathcal{F}_0$ and $\mathcal{F}_2$ were explicitly evaluated in Refs. [14] and [5], respectively, and were found to be nonzero. We do not see how such a relation between τ_{micro} and τ_{eff} could be obtained using the methods of [4], since the modular group ostensibly acts only on τ_{eff} and not on τ_{micro}. Nevertheless, it may be that the series connecting them sums to a modular function, so that the modular group also acts on τ_{micro}.

Of course, the prepotential only controls the leading 2-derivative/4-fermion terms in the gradient expansion along the Coulomb branch of $N = 2$ supersymmetric QCD. In general one has

$$\mathcal{L}_{\text{eff}} \;=\; \mathcal{L}_{\text{2-deriv}} \;+\; \mathcal{L}_{\text{4-deriv}} \;+\cdots \tag{5.13}$$

where

$$\mathcal{L}_{\text{2-deriv}} \;=\; \frac{1}{4\pi}\,\text{Im}\int d^4\theta\,\mathcal{F}(\Psi)\;, \tag{5.14}$$

and

$$\mathcal{L}_{\text{4-deriv}} \;=\; \int d^4\theta d^4\bar{\theta}\,\mathcal{H}(\Psi,\bar{\Psi})\;. \tag{5.15}$$

Here Ψ is the $N = 2$ chiral superfield, and $\mathcal{H}$ is a real function of its arguments [24,25]. The pure n-instanton contribution to $\mathcal{H}$ for pure $SU(2)$ gauge theory, valid to leading semiclassical order, is then given by:

$$\frac{\partial^4}{\partial\bar{\text{v}}^4}\,\mathcal{H}(\text{v},\bar{\text{v}})\,\Big|_{n\text{-inst}} \;=\; 64\pi^8\int d\tilde{\mu}^{(n)}_{\text{phys}}\,e^{-S^0_{\text{inst}}}\sum_{k,k',l,l'=1}^{n}(\nu_k\tau^3 w_k\bar{w}_{k'}\tau^3\nu_{k'})(\mu_l\tau^3 w_l\bar{w}_{l'}\tau^3\mu_{l'})$$
$$\tag{5.16}$$

where the VEVs v and $\bar{\text{v}}$ are to be treated as independent. This expression was written down in Ref. [15]; the new information here is the explicit definition of the collective coordinate measure $d\tilde{\mu}^{(n)}_{\text{phys}}$, defined by Eqs. (5.4), (3.19) and (3.27). For the pure n-antiinstanton contribution, exchange v and $\bar{\text{v}}$, while for $N_F > 0$ make the changes (5.10). As emphasized in [15], in contrast to the nonrenormalized holomorphic prepotential, and excepting the special case $N_F = 4$ [26], there will in general be perturbative corrections to Eq. (5.16) as well as mixed n-instanton, m-antiinstanton contributions to $\mathcal{H}$ not governed by Eq. (5.16).

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We close with a practical comment on the doability of these integrations for the pre-potentials. Notice that when $\mathcal{A}_{\text{tot}}$ is eliminated at the outset in favor of the other moduli via the invertible linear equation (3.7), then the action (5.1) becomes a complicated algebraic expression, and the resulting integrations are quite involved; only numerical methods appear promising for $n > 2$. On the other hand, if $\mathcal{A}_{\text{tot}}$ is treated as an independent integration variable (as we have been doing throughout), then both the n-instanton action (5.1) and (5.8), and the arguments of the constraint δ-functions in (3.19), are quadratic forms in the collective coordinates $\{a, \mathcal{M}, \mathcal{N}, \mathcal{K}, \tilde{\mathcal{K}}\}$. (In fact the action only involves the top-row elements of a, $\mathcal{M}$ and $\mathcal{N}$.) It follows that the above-given expressions for the $\mathcal{F}_n$ are amenable to standard methods of analysis for (supersymmetric) Gaussian integrals. For instance, if one were to exponentiate the constraints in the usual way, by means of a supermultiplet of Lagrange multipliers, then the Gaussian variables $\{a, \mathcal{M}, \mathcal{N}, \mathcal{K}, \tilde{\mathcal{K}}\}$ can be integrated out entirely, and replaced by the appropriate superdeterminant. Only the $\mathcal{A}_{\text{tot}}$ integration, and the integration over the Lagrange multipliers, remains. It would be interesting if such an expression would naturally yield a recursion formula in n for the $\mathcal{F}_n$, such as Matone's [22] for instance.

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